The Dot Product

Students will find the dot product, find the angle between two vectors and determine if vectors are orthogonal.

Precalculus

How do you find the dot product?

Definition of Dot Product:

The dot product of two vectors results in a **number**, *not a vector.*

If **v** = $<a_1, b_1>$ and **w** = $<a_2, b_2>$, then the dot product is:

$$\boxed{\mathbf{v}\cdot\mathbf{w} = a_1a_2 + b_1b_2}$$

Example 1: *Find the Dot Product.*

v = <5, – 2> **w = <-3, 4>**

a. **v•w** $5(-3)+(-2)(4) = -23$

b. **w•v** $(-3)(5)+4(-2) = -23$

c. **v•v** $5(5)+(-2)(-2) = 29$

v = <7, 4> w = <2 ,– 1>

a. **v•w** $7(2)+4(-1) = 10$

b. **w•v** $2(7)+(-1)(4) = 10$

c. **w•w** $(2)(2)+(-1)(-1) = 5$

6.7 The Dot Product

Precalculus

Students will find the dot product, find the angle between two vectors and determine if vectors are orthogonal.

What are the properties of the dot product?

Properties of the Dot Product

1. $u \bullet v = v \bullet u$
2. $u \bullet (v+w) = u \bullet v + u \bullet w$
3. $0 \bullet v = 0$
4. $v \bullet v = ||v||^2$
5. $(cu) \bullet v = c(u \bullet v) = u \bullet (CV)$

How do you find the sngle between two vectors?

Finding the Angle between two Vectors

$$\cos\theta = \frac{v \bullet w}{||v||\,||w||} \quad \text{so} \quad \theta = \cos^{-1}\left(\frac{v \bullet w}{||v||\,||w||}\right)$$

Example 2: Find the angle between the two vectors.

a. $v = \langle 3, 2\rangle$ $w = \langle 1, 4\rangle$

$V \cdot W = 3(1) + 2(4) = 11$

$||V|| = \sqrt{3^2+2^2} = \sqrt{13}$

$||W|| = \sqrt{1^2+4^2} = \sqrt{17}$

$\theta = \cos^{-1}\left(\frac{11}{\sqrt{13} \cdot \sqrt{17}}\right)$

$\theta \approx 42.3°$

b. $v = \langle 4, 3\rangle$ $w = \langle 1, 2\rangle$

$V \cdot W = 4(1) + 3(2) = 10$

$||V|| = \sqrt{4^2+3^2} = \sqrt{25} = 5$

$||W|| = \sqrt{1^2+2^2} = \sqrt{5}$

$\theta = \cos^{-1}\left(\frac{10}{5 \cdot \sqrt{5}}\right)$

$\theta \approx 26.6°$

6.7 The Dot Product

Students will find the dot product, find the angle between two vectors and determine if vectors are orthogonal.

Precalculus

What are orthogonal vectors?

Determining if Vectors Are Parallel, Orthogonal or Neither.

"orthogonal" means perpendicular

To determine if vectors are parallel, orthogonal or neither find the angle between the two vectors.

θ	**Parallel, Orthogonal, or Neither**
0°	Parallel (Going the same direction)
180°	Parallel (Going opposite directions)
90°	Orthogonal
other	Neither

Example 3: Determine if the vectors are Parallel, Orthogonal or Neither.

a. v = <6, -3> w = <1, 2>

$V \cdot W = 6(1) + (-3)(2) = 0$

$\|V\| =$

$\|W\| =$ X

$\theta = \cos^{-1}\left(\frac{0}{?}\right) = \cos^{-1}(0)$

$\theta = 90°$ orthogonal

b. v = <4, -3> w = <-8, 6>

$V \cdot W = 4(-8) + (-3)(6) = -50$

$\|V\| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$

$\|W\| = \sqrt{(-8)^2 + 6^2} = \sqrt{100} = 10$

$\theta = \cos^{-1}\left(\frac{-50}{5 \cdot 10}\right)$

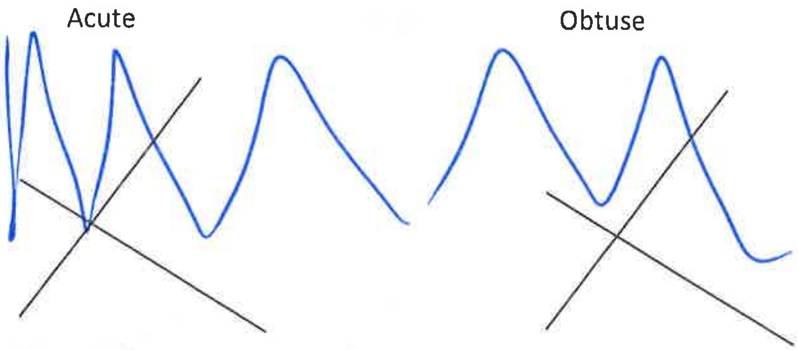
$\theta = 180$ parallel, opposite directions

6.7 The Dot Product Day 2

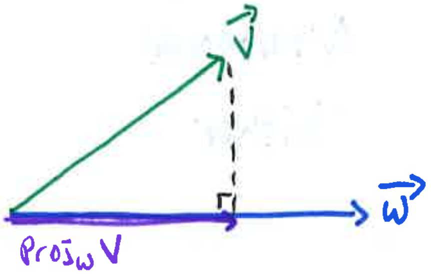
Students will find the projection of one vector onto another, express a vector as the sum of two orthogonal vectors and compute work

Precalculus

Projection of a vector onto another vector



Projection of v onto ω $(\text{Proj}_\omega v)$



$$\text{Proj}_\omega v = \frac{v \cdot \omega}{\|\omega\|^2} \cdot \omega$$

Example 4: Find the vector projection of **v** onto **w**.

$||w|| = \sqrt{(-2)^2 + 6^2}$

$= \sqrt{40}$

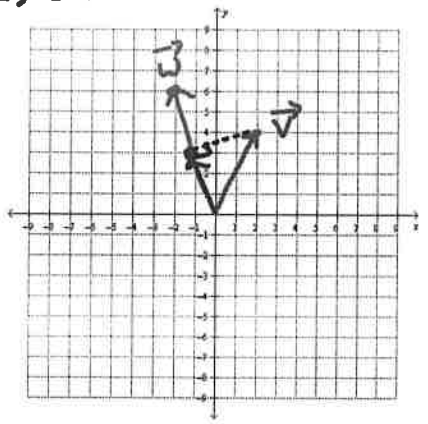
$v \cdot w = 2(-2) + 4(6) = 20$

v = <2,4> w = <-2,6>

$\text{Proj}_w V = \frac{20}{\sqrt{40}^2} \cdot \langle -2, 6 \rangle$

$= \frac{20}{40} \cdot \langle -2, 6 \rangle$

$\text{Proj}_w V = \langle -1, 3 \rangle$



6.7 The Dot Product Day 2

Students will find the projection of one vector onto another, express a vector as the sum of two orthogonal vectors and compute work.

Precalculus

Example 4: Find the vector projection of **v** onto **w**.

v = <2,4> **w** = <-2,6>

The vector components of v

6.7 The Dot Product Day 2

Students will find the projection of one vector onto another, express a vector as the sum of two orthogonal vectors and compute work

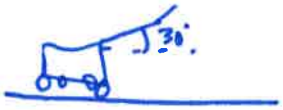
Precalculus

Example 5: Decompose **v** into vectors **v**$_1$ and **v**$_2$, where **v**$_1$ is parallel to **w** and **v**$_2$ is orthogonal to **w.**

Definition of Work

$$W = \vec{F} \cdot \cos\theta \cdot \vec{d}$$

Example 6: A child pulls a wagon along level ground by exerting a force of $\boxed{\overset{F}{20}}$ pounds on the handle that makes a $\underset{\theta}{\boxed{30°}}$ angle with the horizontal. How much work is done pulling the wagon $\underset{d}{\underline{150}}$ feet?



$$W = 20 \cdot \cos(30°) \cdot 150$$

$$W = 2598 \text{ ft. lbs.}$$

Dot Product of Vectors Day 1

Name:_______________ Date:______

Find the dot product of **u** *and* **v**.

1. **u** = ⟨4, 5⟩ **v** = ⟨-3, -7⟩

$u \cdot v = 4(-3) + 5(-7) = -47$

2. **u** = ⟨2, -4⟩ **v** = ⟨-8, 7⟩

$u \cdot v = 2(-8) + (-4)(7) = -44$

3. **u** = ⟨-2, 7⟩ **v** = ⟨-5, -8⟩

$u \cdot v = (-2)(-5) + 7(-8) = -46$

4. **u** = ⟨-5, 2⟩ **v** = ⟨8, 13⟩

$u \cdot v = (-5)(8) + 2(13) = -14$

Find the angle θ between the vectors.

5. a. **u** = ⟨-4, -3⟩**v** = ⟨-1, 5⟩

$u \cdot v = (-4)(-1) + (-3)(5) = -11$

$\|u\| = \sqrt{(-4)^2 + (-3)^2} = 5$

$\|v\| = \sqrt{(-1)^2 + 5^2} = \sqrt{26}$

$\theta = \cos^{-1}\left(\frac{-11}{5 \cdot \sqrt{26}}\right) \approx 115.56°$

b. **u** = ⟨2, -2⟩ **v** = ⟨-3, 3⟩

$u \cdot v = 2(-3) + (-2)(3) = -12$

$\|u\| = \sqrt{2^2 + (-2)^2} = \sqrt{8}$

$\|v\| = \sqrt{(-3)^2 + 3^2} = \sqrt{18}$

$\theta = \cos^{-1}\left(\frac{-12}{\sqrt{8} \cdot \sqrt{18}}\right) \approx 180°$

6. a. **u** = ⟨2, 3⟩ **v** = ⟨-3, 5⟩

$u \cdot v = 2(-3) + 3(5) = 9$

$\|u\| = \sqrt{2^2 + 3^2} = \sqrt{13}$

$\|v\| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$

$\theta = \cos^{-1}\left(\frac{9}{\sqrt{13} \cdot \sqrt{34}}\right) \approx 64.65°$

b. **u** = ⟨-3, -3⟩ **v** = ⟨2, 2√3⟩

$u \cdot v = (-3)(2) + (-3)(2\sqrt{3}) = -6 - 6\sqrt{3} \approx -16.39$

$\|u\| = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18}$

$\|v\| = \sqrt{2^2 + (2\sqrt{3})^2} = 4$

$\theta = \cos^{-1}\left(\frac{-16.39}{\sqrt{18} \cdot 4}\right) \approx 165°$

Dot Product Day 2

Name:_______________

In exercises 1 – 4, find the vector projection of **u** onto **v**. ~~Then write u as a sun of two orthogonal vectors, one of which is $\text{proj}_v u$~~.

1. $u = \langle -8, 3\rangle$, $v = \langle -6, -2\rangle$

$$u \cdot v = (-8)(-6) + 3(-2) = 42$$
$$\|v\| = \sqrt{(-6)^2 + (-2)^2} = \sqrt{40}$$
$$\text{Proj}_v u = \frac{42}{(\sqrt{40})^2} \cdot \langle -6, -2\rangle$$
$$= \frac{42}{40} \cdot \langle -6, -2\rangle$$
$$= \left\langle \frac{-63}{10}, \frac{-21}{10} \right\rangle$$

2. $u = \langle 3, -7\rangle$, $v = \langle -2, -6\rangle$

$$u \cdot v = 3(-2) + (-7)(-6) = 36$$
$$\|v\| = \sqrt{(-2)^2 + (-6)^2} = \sqrt{40}$$
$$\text{Proj}_v u = \frac{36}{(\sqrt{40})^2} \cdot \langle -2, -6\rangle$$
$$= \frac{36}{40} \cdot \langle -2, -6\rangle$$
$$= \left\langle \frac{-9}{5}, \frac{-27}{5} \right\rangle$$

3. $u = \langle 8, 6\rangle$, $v = \langle -9, -2\rangle$

$$u \cdot v = 8(-9) + 6(-2) = -84$$
$$\|v\| = \sqrt{(-9)^2 + (-2)^2} = \sqrt{85}$$
$$\text{Proj}_v u = \frac{-84}{(\sqrt{85})^2} \cdot \langle -9, -2\rangle$$
$$= \frac{-84}{85} \cdot \langle -9, -2\rangle$$
$$= \left\langle \frac{756}{85}, \frac{168}{85} \right\rangle$$

4. $u = \langle -2, 8\rangle$, $v = \langle 9, -3\rangle$

$$u \cdot v = (-2)(9) + 8(-3) = -42$$
$$\|v\| = \sqrt{9^2 + (-3)^2} = \sqrt{90}$$
$$\text{Proj}_v u = \frac{-42}{(\sqrt{90})^2} \cdot \langle 9, -3\rangle$$
$$= \frac{-42}{90} \cdot \langle 9, -3\rangle$$
$$= \left\langle \frac{-21}{5}, \frac{7}{5} \right\rangle$$

5. Ojemba is sitting on a sled on the side of a hill inclined at 60°. The combined weight of Ojemba and the sled is 160 pounds. What is the magnitude of the force required for Mandisa to keep the sled from sliding down the hill?

$$V: \langle \cos 60°, \sin 60° \rangle$$