Pre-Calculus Notes

Key Features of Rational Functions

Name KEY

Date______________Per______

What is a rational function?

If *N(x)* and *D(x)* are functions with *D(x)* ≠0, $f(x) = \frac{N(x)}{D(x)}$ ***is a rational function.***

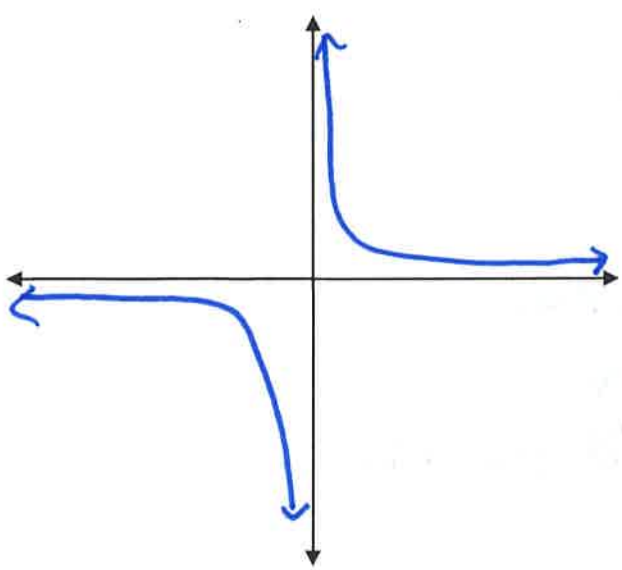
First let's look at the most simple form of a rational function $f(x) = \frac{1}{x}$: .

Sketch the graph of $f(x) = \frac{1}{x}$ by filling out the table of values below.

x	2	1	0.5	0.1	0.01	0.001	→0
$f(x)$	$\frac{1}{2}$	1	2	10	100	1,000	∞

Show graph on Nspire?

x	-2	-1	-0.5	-0.1	-0.01	-0.001	→0
$f(x)$	$-\frac{1}{2}$	-1	-2	-10	-100	-1,000	$-\infty$



Use the graph to determine ***domain*** and ***range.***

Domain: $(-\infty,0)\cup(0,\infty)$

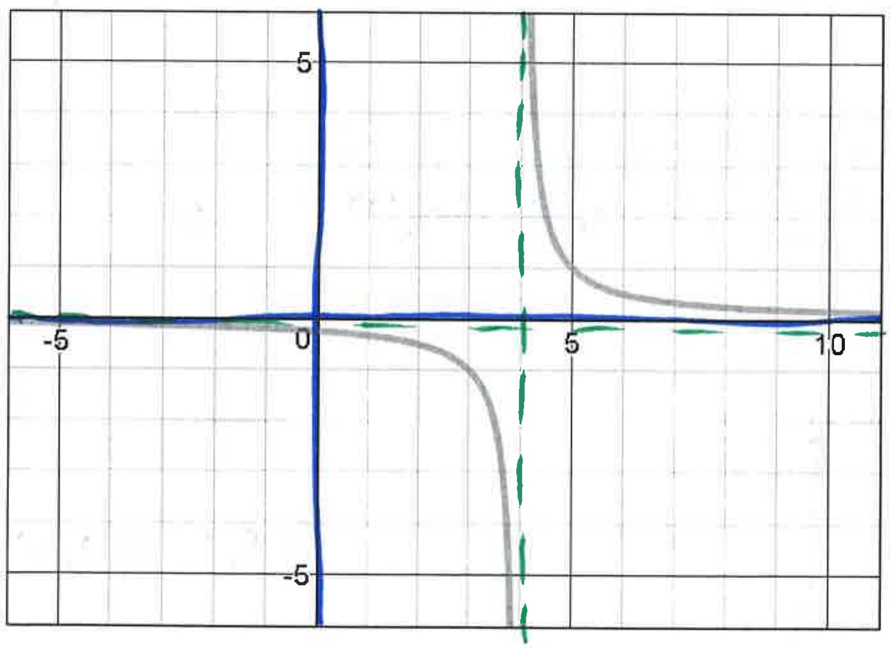
Range: $(-\infty,0)\cup(0,\infty)$

When analyzing rational functions, we look for specific key features. These are, x- and y-intercepts, and asymptotes.

Two basic types of asymptotes:

- Vertical
  - Indicates a restriction on the domain of a function
  - $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$
- End Behavior: Can be horizontal, slant or a non-linear function
  - Indicates a restriction on the range of a function
  - $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$

Example 1: Find the key features of the rational function graphed below.



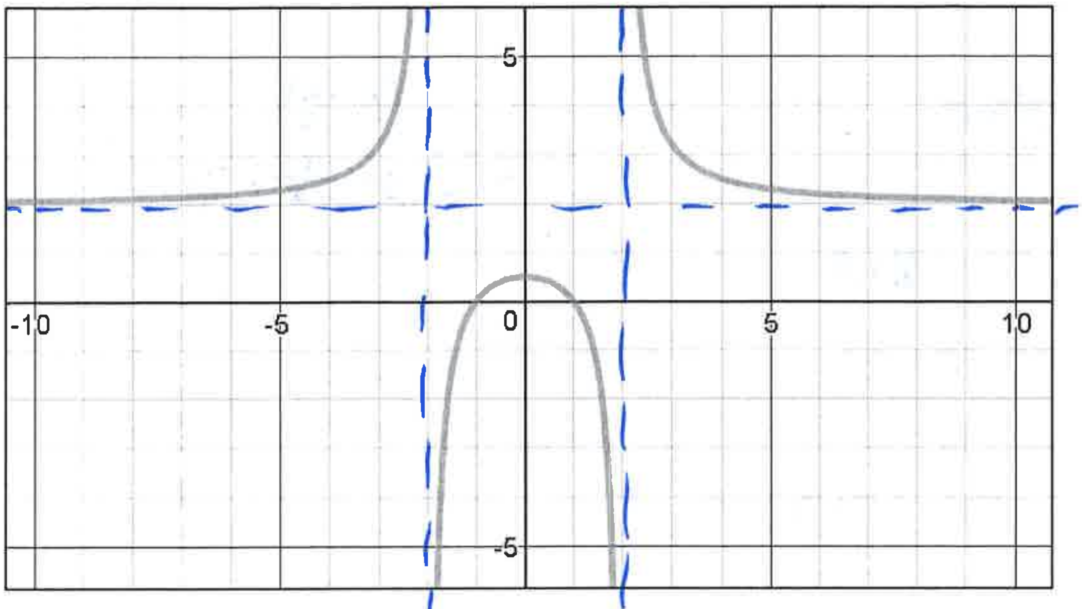
x-intercept(s) – where does it intersect the x-axis? None

y-intercept – where does it intersect the y-axis? $-\frac{1}{4}$ (estimated) $(0, -\frac{1}{4})$

vertical asymptote(s) – $x = 4$

end behavior asymptote - $y = 0$

Example 2: Find the key features of the rational function graphed below.



x-intercept(s) -1 and 1
(-1,0) (1,0)

y-intercept $(0, \frac{1}{2})$

vertical asymptote(s) – X = -2, x = 2

end behavior asymptote - y = 2

*day 2 ***Now, what if you don't have the graph, you only have the equation?***

Always factor first if you can

x-intercepts: (a.k.a. "solutions") Occur at the zeros of the numerator (set numerator = 0 and solve).

y-intercept: Occurs when x = 0 (plug 0 in for x and simplify).

Vertical: Occur at zeros of the denominator (set denominator = 0 and solve)

End Behavior: Compare the degrees of the numerator (n) and denominator (d)

- If n < m, the asymptote is horizontal; $y = 0$
- If n = m, the asymptote is horizontal; $y = \frac{a_n}{b_m}$
- If n > m, end behavior asymptote is found by actually dividing the polynomials (using long or synthetic division)

$n=0$

Example 3: Find the key features of the rational function, $f(x) = \frac{4}{(x-2)^3}$. $m=3$

x-intercept(s) – None

y-intercept – $(0, \frac{-1}{2})$

$f(x) = \frac{4}{(0-2)^3} = \frac{4}{-8} = \frac{-1}{2}$

vertical asymptote(s) – $x=2$

$(x-2)^3=0$

$x-2=0$

end behavior asymptote - $y=0$

$n<m$

Example 4: Find the key features of the rational function, $f(x) = \frac{2x^{2}}{(x^{2}-4)}$. n, m

x-intercept(s) – $(0,0)$

$2x^2=0$

$x^2=0$

y-intercept – $(0,0)$

$f(x) = \frac{2(0)^2}{(0^2-4)} = \frac{0}{-4}$

vertical asymptote(s) – $x=-2$, $x=2$

$(x^2-4)=0$

$x^2=4$

$x=\pm 2$

end behavior asymptote - $y=2$

$n=m$ $\quad y=\frac{2}{1}$

Example 7: $f(x) = \frac{x^2+x-6}{x-3} = \frac{(x+3)(x-2)}{x-3}$

$$\begin{array}{r|rrr} 3 & 1 & 1 & -6 \\ & \downarrow & 3 & 12 \\ \hline & 1 & 4 & 6 \end{array}$$

X-int: ~~x~~ $(-3,0)$ $(2,0)$

$x+3=0 \quad x-2=0$

y-int: $(0,2)$

$f(0) = \frac{0+0-6}{0-3} = \frac{-6}{-3}$

Vert: $x=3$

$x-3=0$

End behavior: $y=x+4$

$n=2$
$m=1$ $\quad n>m$

Example 5: Find the key features of the rational function, $f(x) = \frac{2x^2}{x+1}$. n

x-intercept(s) – (0, 0)

$2x^2=0$

$x^2=0$

y-intercept – (0,0)

$f(x)=\frac{2(0)^2}{0+1}=\frac{0}{1}$

vertical asymptote(s) – $x=-1$

$x+1=0$

$x=-1$

end behavior asymptote - Just this $\boxed{y=2x-2}$ $\frac{2}{x+1}$ don't need for asymptote

n=2 n>m

m=1

$$\begin{array}{r}
2x-2 \\
x+1\overline{)2x^2+0x+0} \\
-(2x^2+2x) \\
\hline
-2x+0 \\
-(-2x-2) \\
\hline
2
\end{array}$$

OR synthetic

$$\begin{array}{r|rrr}
-1 & 2 & 0 & 0 \\
 & \downarrow & -2 & 2 \\
\hline
 & 2 & -2 & 2
\end{array}$$

Remainder

$y=2x-2$

Example 6: Find the key features of the rational function, $f(x) = \frac{x^3-3x^2+3x+1}{x-1}$.

must Graph on NSpire... we won't do that on the test :)

x-intercept(s) – (-0.26, 0)

y-intercept – (0, -1)

$f(x)=\frac{0^3-3(0)^2+3(0)+1}{0-1}=\frac{1}{-1}$

vertical asymptote(s) – $X=1$

$X-1=0$

end behavior asymptote - $y=x^2-2x+1$ $\cancel{\frac{2}{x-1}}$ *Show graph!

n=3 n>m

m=1

$$\begin{array}{r}
x^2-2x+1 \quad r2 \\
x-1\overline{)x^3-3x^2+3x+1} \\
-(x^3-x^2+0+0) \\
\hline
-2x^2+3x+1 \\
-(-2x^2+2x+0) \\
\hline
x+1 \\
-(x-1) \\
\hline
2
\end{array}$$

OR

$$\begin{array}{r|rrrr}
1 & 1 & -3 & 3 & 1 \\
 & \downarrow & 1 & -2 & 1 \\
\hline
 & 1 & -2 & 1 & 2
\end{array}$$

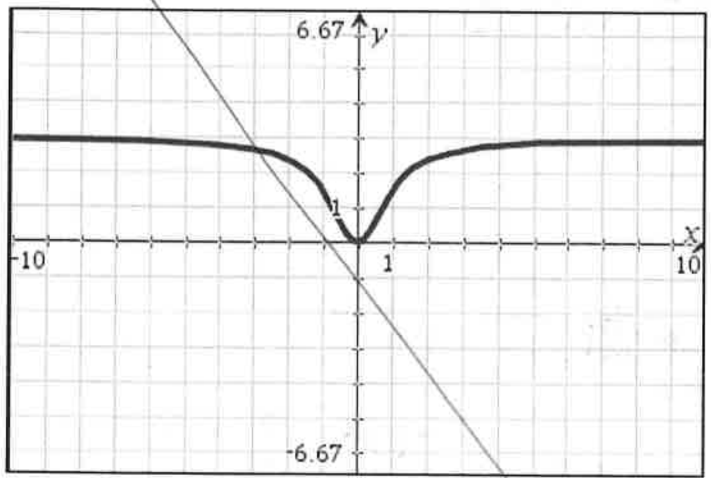
x^2-2x+1 Remainder

Pre-Calculus: Key Features of Rational Functions **Name** ____________________

Day 1

Find all key features of the following functions:

1.



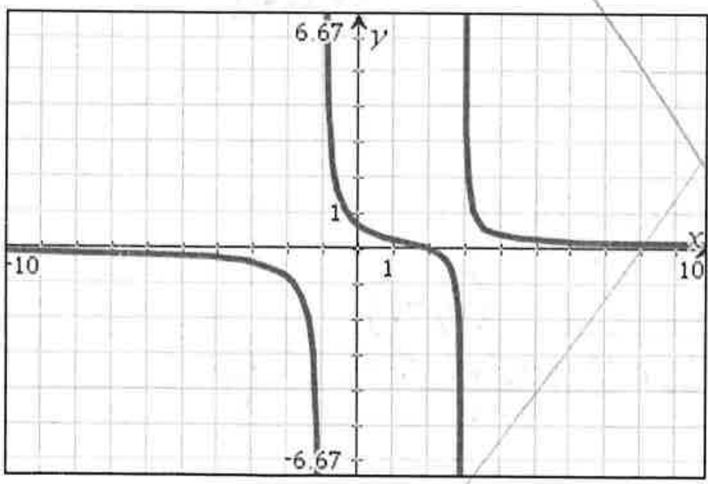
x-intercept(s) __________ y-intercept __________

Vertical Asymptote(s) ____________________

End Behavior ____________________

Domain __________ Range __________

2.



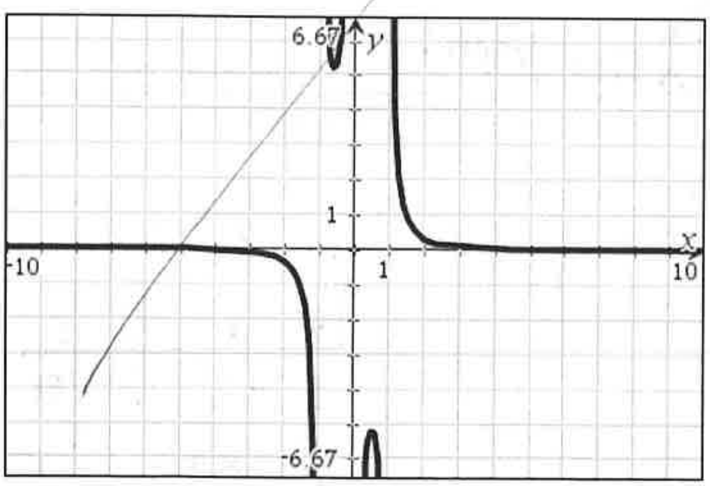
x-intercept(s) __________ y-intercept __________

Vertical Asymptote(s) ____________________

End Behavior ____________________

Domain __________ Range __________

3.



x-intercept(s) __________ y-intercept __________

Vertical Asymptote(s) ____________________

End Behavior ____________________

Domain __________ Range __________

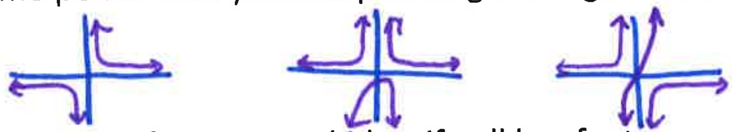
HPC/RPC Graphing Rational Functions NOTES Name________________________________

Date_______________________Per_______

Day 3

The steps given below are going to help with graphing rational functions.

1) Identify x- and y-intercepts (if there are any).
2) Find the vertical asymptote(s) by setting the denominator equal to 0 and solving.
3) Identify any holes in the graph (factors in the denominator that reduce out).
4) Find the end behavior asymptote (if it exists). $n<m \Rightarrow y=0$ $n>m \rightarrow$ divide polynomials
5) Graph asymptotes using dashed lines. $n=m \Rightarrow$ divide coefficients
6) Pick some points that you can plot to get the general shape of the graph.

In general:  These are the basic shapes...

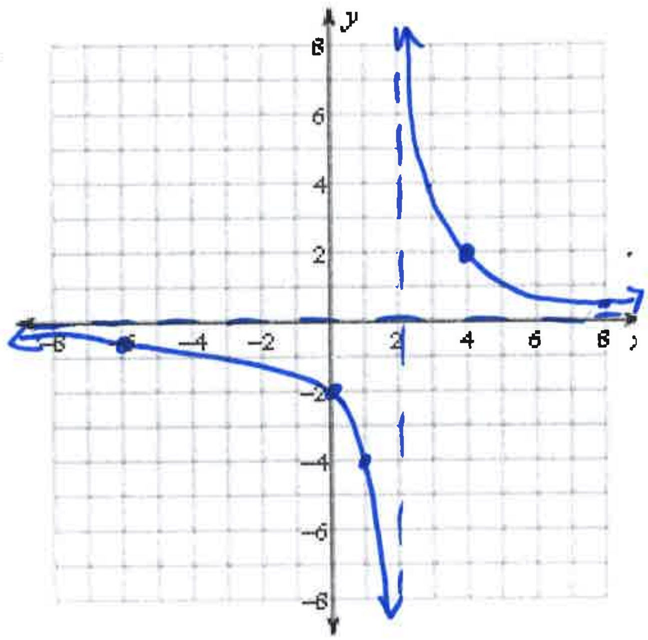
Example 1: Graph the function and identify all key features of the graph.

$f(x) = \frac{4}{x-2}$

1) x-intercept(s)? None $4 \neq 0$

y-intercept? (0, -2)

$f(x) = \frac{4}{0-2} = \frac{4}{-2}$



2) Vertical Asymptote(s) X = 2

X-2=0

3) Holes? None

~~X+2=0~~

4) End Behavior Asymptote y=0

n=0
M=1
n<m

x	f(x)
0	-2
-6	$\frac{4}{-6-2} = \frac{-1}{2}$
1	$\frac{4}{1-2} = -4$
4	$\frac{4}{4-2} = 2$
8	$\frac{4}{8-2} = 4/6$

Example 2: Graph the function and identify all key features of the graph.

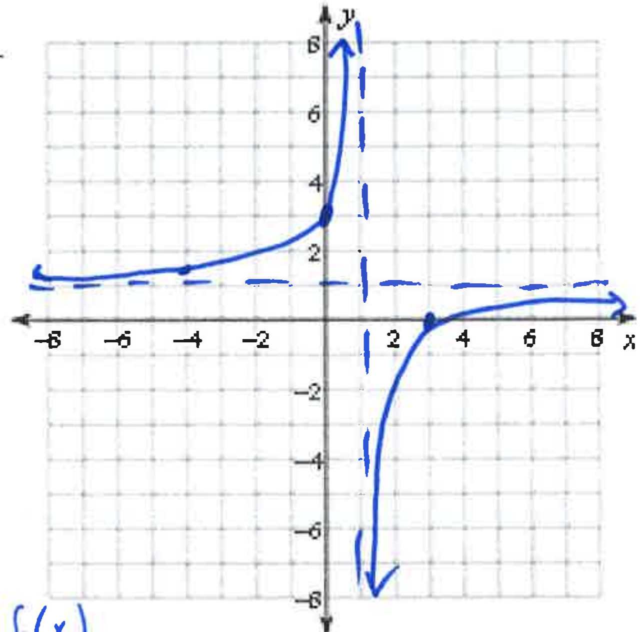
$f(x) = -\frac{2}{x-1} + 1$ *Save for the next day* $\frac{-2}{x-1} + \frac{x-1}{x-1} = \frac{x-3}{x-1}$

1) x-intercept(s)? (3,0)

$x - 3 = 0$

y-intercept? (0,3)

$f(x) = \frac{0-3}{0-1} = 3$



2) Vertical Asymptote(s) $x = 1$

$x - 1 = 0$

3) Holes? None

x	f(x)
0	3
-4	$\frac{-7}{-5} = 1.4$

4) End Behavior Asymptote $y = \frac{1}{1}$ $y = 1$

n:1
m:1 n=m

Example 3: Graph the function and identify all key features of the graph.

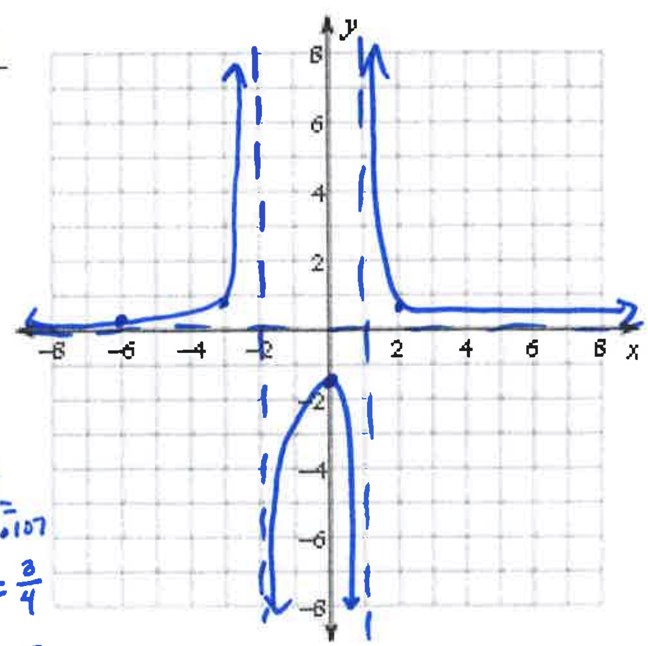
$f(x) = \frac{3}{x^2+x-2}$

1) x-intercept(s)? None

$3 \neq 0$

y-intercept? (0,-1.5)

$f(x) = \frac{3}{0^2+0-2} = \frac{3}{-2}$



2) Vertical Asymptote(s) $x = -2, \; x = 1$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

x	f(x)
-6	$\frac{3}{36-6-2} = .107$
-3	$\frac{3}{9-3-2} = \frac{3}{4}$
2	$\frac{3}{4+2-2} = \frac{3}{4}$

3) Holes? None

n=0 m:2 n<m

4) End Behavior Asymptote $y = 0$

Example 4: Graph the function and identify all key features of the graph.

$$f(x) = \frac{x^2+5x+4}{-2x^2-6x} = \frac{(x+4)(x+1)}{-2x(x+3)}$$

1) x-intercept(s)? (-4,0) (-1,0)

$x^2+5x+4=0$

$(x+4)(x+1)=0$

y-intercept? None

$f(x) = \frac{0^2+5(0)+4}{-2(0^2)-6(0)} = \frac{4}{0}$

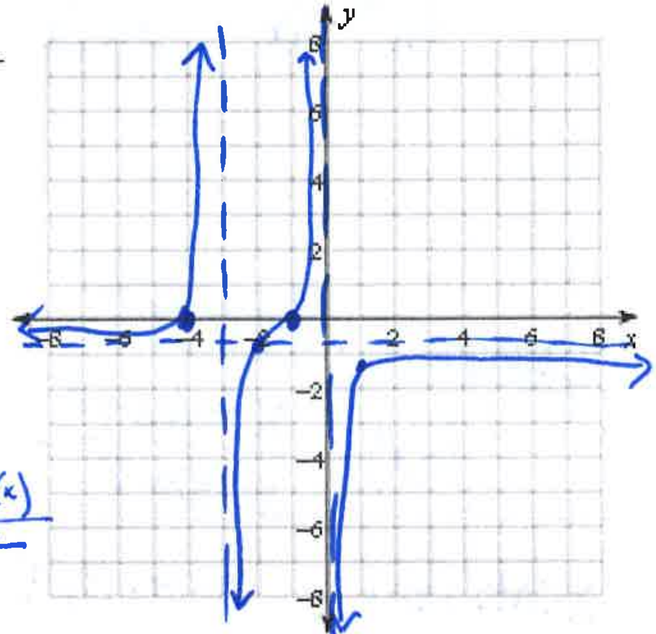
2) Vertical Asymptote(s) $x=0$ $x=-3$

$-2x^2-6x=0$

$-2x(x+3)=0$

$x=0$ $x=-3$

3) Holes? None



x	f(x)
-2	$\frac{-2}{4}$
-4	0
-1	0
1	$\frac{10}{-8} = -1.25$

$n=2$ $m=2$

4) End Behavior Asymptote $y = \frac{1}{-2} = -\frac{1}{2}$

Example 5: Graph the function and identify all key features of the graph.

$$f(x) = \frac{x^3+x^2-12x}{3x^2+3x} = \frac{x(x^2+x-12)}{3x(x+1)} = \frac{x(x+4)(x-3)}{3x(x+1)}$$ → x's simplify

1) x-intercept(s)? (-4,0) (3,0)

$x+4=0$ $x-3=0$

$x=-4$ $x=3$

y-intercept? None

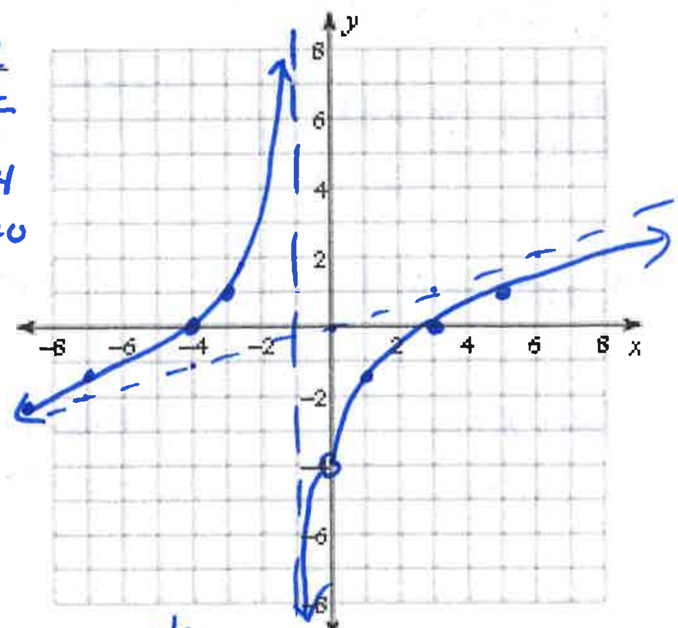
$f(x) = \frac{(0+4)(0-3)}{3(0+1)} = \frac{-12}{3} = -4$

But there's a hole @ $x=0$

2) Vertical Asymptote(s) $x=-1$

$3x(x+1)=0$ $\cancel{x=0}$ $x=-1$

3) Holes? $x=0$



x	f(x)
-9	-2.5
-7	-1.6
-3	1
1	-1.6
5	1

$$\begin{array}{r|l} & \frac{1}{3}x \\ 3x^2+3x & x^3+x^2-12x+0 \\ & -(x^3+x^2) \\ & -12x \end{array}$$

$n=3$ $m=2$ $n>m$

4) End Behavior Asymptote $y=\frac{1}{3}x$ slant asymptote

Example : $f(x) =$

$$f(x) = \frac{x^3 - 16x}{3x^2 - 9x} = \frac{x(x^2-16)}{3x(x^2-9)} = \frac{x(x+4)(x-4)}{3x(x+3)(x-3)}$$

Example 5 : Hole example

$$f(x) = \frac{x^2 - 6x + 8}{x^2 - 5x + 6} = \frac{(x-4)\cancel{(x-2)}}{(x-3)\cancel{(x-2)}}$$

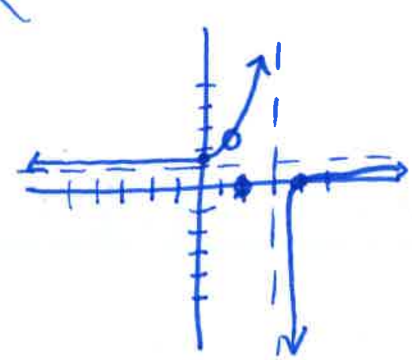
Hole @ $x - 2 = 0$
$x = 2$

x-int: $(4, 0)$ y-int: $(0, 4/3)$

Vert: $x = 3$ E.B. $n = 2$, $m = 2$ $y = 1$

Hole: $(2, 2)$

$\frac{2-4}{2-3} = \frac{-2}{-1} = 2$



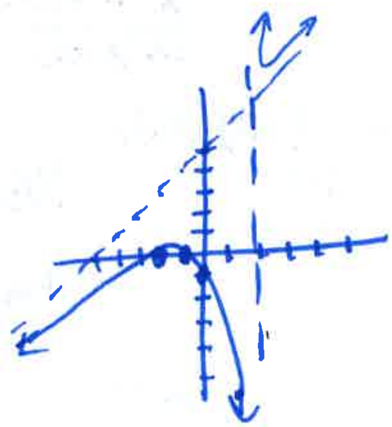
Division example

$$f(x) = \frac{x^2 + 3x + 2}{x - 2} = \frac{(x+2)(x+1)}{x-2}$$

x-int: $(-2, 0)$ $(-1, 0)$ y-int: $(0, -1)$

Vert: $x = 2$ E.B. $n = 2$, $m = 1$ $n > m$ ÷

$y = x + 5$



2	1	3	2
	↓	2	10
	1	5	12

$f(1) = \frac{3(2)}{-1} = -6$

$f(3) = \frac{5(4)}{1} = 20$