Logistic Functions

Analyze & Model problems using Logistic Functions

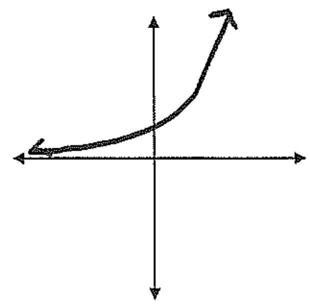
Construct & Solve Logistic Functions; Std. #3 F-BF.5

HPC/RPC

KEY

What is a Logistic Function?

Exponential growth (from $f(x)=ab^x$) is unrestricted (No upper bound)

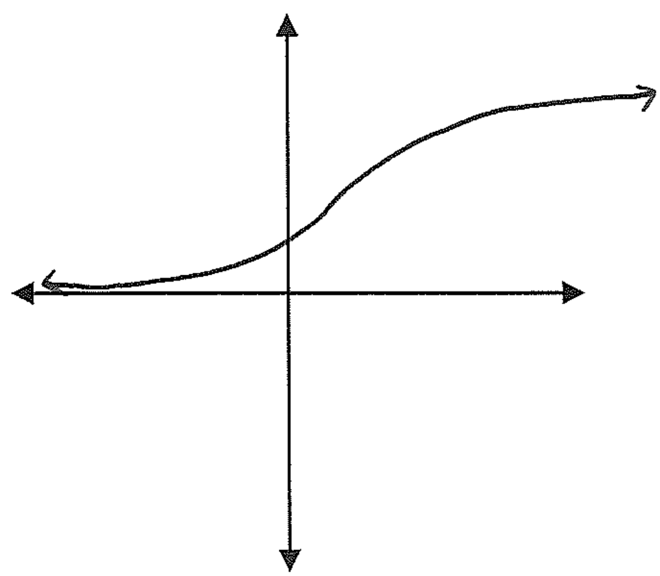


BUT:

- **Plants can only grow so big**
- **Only so many goldfish can fit into a bowl**

So, some growth situations have limits. They begin exponentially, but level out over time.

Logistic Functions (graphically)



What does a logistic function look like?

Logistic growth/decay formulas:

$$f(x)=\frac{c}{1+a\cdot b^x} \quad \text{or} \quad f(x)=\frac{c}{1+a\cdot e^{-kx}}$$

- c is the limit to growth (a.k.a. "maximum sustainable growth") *horizontal asymptote*
- a, b and c are positive constants
- If $b > 1$ (or $k > 0$), the function represents logistic growth
- If $b < 1$, (or $k < 0$), the function represents logistic decay

Determine the horizontal asymptotes and the y-intercept for each:

a. $f(x)=\frac{8}{1+3(0.7)^x}$

Asymptotes:
$y=8$ and $y=0$

y-intercept: $(x=0)$

$y=\frac{8}{1+3(0.7)^0}$

$=\frac{8}{1+3}$

$=\frac{8}{4}$

$=2$ $(0,2)$

b. $g(x)=\frac{20}{1+2e^{-3x}}$

Asymptotes:
$y=20$ and $y=0$

y-intercept:

$y=\frac{20}{1+2e^{-3(0)}}$

$=\frac{20}{1+2}$

$=\frac{20}{3}\approx 6.67$ $(0,6.67)$

Exponential and Logistic Modeling Name______________________________

1. On a college campus of 5000 students, one student returns from vacation with a contagious flu virus. The spread of the virus is modeled by $y = \dfrac{5000}{1 + 4999e^{-0.8t}}$, $t > 0$, where y is the total number of students infected after t days. The college will cancel classes when 40% or more of the students are infected.

a. How many students are infected after 5 days? $t(5)$

$y = \dfrac{5000}{1+4999e^{-0.8(5)}} = 54$ people

b. After how many days will the college cancel classes? (40% of 5000 = 2000)

$2000 = \dfrac{5000}{1+4999e^{-0.8t}}$

$2000(1+4999e^{-0.8t}) = 5000$

$1+4999e^{-0.8t} = \frac{5}{2}$ -1 -1

$4999e^{-0.8t} = 3/2$

$\rightarrow$ ln $e^{-0.8t} = \dfrac{3}{\ln 9998}$

$\dfrac{-0.8t}{-0.8} = \dfrac{-8.11153}{-0.8}$

$\boxed{t = 10 \text{ days (approx.)}}$

2. Newton's Law of Cooling is represented by the equation $T = C + (T_0 - C)e^{-kt}$, where

T = "final" temperature of a heated object
C = constant temperature of the surrounding medium (the ambient temperature)
T_0 = initial temperature of the heated object
k = negative constant associated with the cooling object (unique to each scenario)
t = time (in minutes)

A pizza is taken from a 425°F (T_0) oven and placed on the counter to cool. If the temperature in the kitchen is 75°F (C) and the cooling rate for this type of pizza is $k = 0.35$,

a. What is the temperature (to the nearest degree) of the pizza after 2 minutes?

$T = 75 + (425-75)e^{-0.35(2)}$ $T = 249°$

b. To the nearest minute, how long until the pizza has cooled to a temperature below 90°F?

$90 = 75 + (350)e^{-0.35t}$

$15 = 350e^{-0.35t}$

$\frac{3}{70} = e^{-0.35t}$

$\rightarrow \ln\left(\frac{3}{70}\right) = -0.35t$

$-3.1499 = -0.35t$

$\boxed{9 \text{ minutes} = t}$

c. If Matt and Tyler like to eat their pizza at a temperature of about 110°F, now many minutes should they wait before they "dig in"?

$110 = 75 + 350e^{-0.35t}$

$35 = 350e^{-0.35t}$

$\frac{1}{10} = e^{-0.35t}$

$\ln\left(\frac{1}{10}\right) = -0.35t$

$\dfrac{-2.303}{-0.35} = \dfrac{-0.35t}{-0.35}$

About 7 minutes

3. Newton's Law of Cooling applies equally well if the "cooling is negative", meaning the object is taken from a colder medium and placed in a warmer one. If a can of soda is taken from a 35°F cooler and placed in a room where the temperature is 75°F, how long will it take the drink to warm up to 65°F? Use 0.031 for *k*.

$65 = 75 + (35-75)e^{-0.031t}$

$-10 = -40e^{-0.031t}$

$\frac{1}{4} = e^{-0.031t}$

$\ln\left(\frac{1}{4}\right) = -0.031t$

$-1.386 = -0.031t$

$\boxed{\text{About 45 minutes} = t}$

4. Wood products are classified according to their life span. There are four classifications: short (1 year), medium short (4 years), medium long (16 years) and long (50 years). The percentage of remaining wood products after *t* years for wood products with long life spans is given by $y = \dfrac{100.3952}{1+0.0316e^{0.0581t}}$

a. What is the decay rate?

b. What is the percentage of wood products remaining after 10 years?

$y = \dfrac{100.3952}{1+0.0316e^{0.0581(10)}}$ $y = 95\%$

c. How long does it take for the percentage of remaining wood products to reach 50%?

$50 = \dfrac{100.3952}{1+0.0316e^{0.0581t}}$

$1+0.0316e^{0.0581t} = \dfrac{100.3952}{50}$

$1+0.0316e^{0.0581t} = 2.0079$

$0.0316e^{0.0581t} = 1.0079$

$e^{0.0581t} = 31.896$

$\ln 31.896 = 0.0581t$

$\boxed{\text{About 60 years} = t}$

d. Explain why the numerator given in the model is reasonable.

5. A hard-boiled egg at a temperature of 96°C (T_0) is placed in 16°C (C) water to cool. Four minutes later the temperature of the egg is 45°C. Use Newton's Law of Cooling to determine when the egg will be 20°C.

$45 = 16 + (96-16)e^{-k(4)}$

$29 = 80e^{-4k}$

$\frac{29}{80} = e^{-4k}$

$\ln\left(\frac{29}{80}\right) = -4k$

$-1.01473 = -4k$

$0.254 = k$

$20 = 16 + (96-16)e^{-0.254t}$

$4 = 80e^{-0.254t}$

$\frac{1}{20} = e^{-0.254t}$

$\ln\left(\frac{1}{20}\right) = -0.254t$

About 12 minutes

6. Fruit flies are placed in a half-pint milk bottle with a banana (for food) and yeast plants (for food and to provide a stimulus to lay eggs). Suppose that the fruit fly population after t days is modeled by

$$y = \frac{230}{1 + 56.5e^{-0.37t}}$$

a. What is the maximum capacity of the milk bottle and the growth rate of the fruit flies?

max: 230 flies Growth rate:

b. Determine the initial population.

$t(0) = \frac{230}{1+56.5e^{-0.37(0)}} = \frac{230}{57.5} = 4$ flies

c. What is the population after 5 days?

$y = \frac{230}{1+56.5e^{-0.37(5)}}$ $y = 23$ flies

d. How long does it take the population to reach 180?

$180 = \frac{230}{1+56.5e^{-0.37t}}$

$1+56.5e^{-0.37t} = \frac{230}{180}$

-1 -1

$56.5e^{-0.37t} = \frac{5}{18}$

$e^{-0.37t} = 0.004916$

$-0.37t = \ln(0.004916)$

$\boxed{t = 14 \text{ days}}$

e. How long does it take for the population to reach half the maximum capacity?

$115 = \frac{230}{1+56.5e^{-0.37t}}$

$1+56.5e^{-0.37t} = \frac{230}{115}$

$56.5e^{-0.37t} = 2$

$e^{-0.37t} = 0.035398$

$-0.37t = \ln(0.035398)$

$\boxed{9 \text{ days} = t}$

7. Teresa was late getting ready for a party, and the liters of soft drinks she bought were still at room temperature (73°F) with guests due to arrive in 15 minutes. If she puts the bottles in her freezer at -10°F, will the drinks be cold enough (35°F) by the time her guests arrive? Assume $k = 0.031$.

$35 = -10 + (73 - -10)e^{-0.031t}$

$45 = 83e^{-0.031t}$

$\frac{45}{83} = e^{-0.031t}$

$\ln\left(\frac{45}{83}\right) = -0.031t$

$19.7 = t$

It will take almost 20 minutes, so her guests will need to wait for a little bit :)